Journal of Statistical Physics, Vol. 72, Nos. 3/4, 1993

Diffusive Transport Enhancement and Escape Processes in Frictionless Nonlinear Oscillators with Noise

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Received May 15, 1992; final April 5, 1993

The time-dependent escape rates and evolution of a distribution density are considered for a Hamiltonian many-dimensional nonlinear oscillator with external noise. The Hamiltonian dynamics is assumed to be nearly integrable and is described in terms of isolated nonlinear resonances. In case of a small angle between the resonant oscillations and the resonance line, a dynamic enhancement of diffusion occurs inside the separatrix, leading to a strongly enhanced growth of distribution tails and escape rates even when the resonances are relatively narow. The underlying mechanism of the phenomenon is essentially many-dimensional.

KEY WORDS: Nonlinear resonance; distribution function; weak-noise asymptotics.

1. INTRODUCTION

The study of dynamical systems (represented by ordinary differential equations) under the influence of noise constitutes a significant part of the activity of statistical physics. A particular class of problems of this kind is the set of large-deviation (or rare events) problems (see, e.g., ref. 1 for review). In this category fit all the so-called "escape rate" problems, where the phase space of a deterministic system is decomposed into several basins of attraction, with attractors being stable points or limit cycles. The quantity of interest there is the rate of escape of particles, under the influence of noise, from one attractor to another or to an absorbing boundary.[1]

In the present paper, we consider the same distribution-tails/escape-rate problem in quite a different situation when the deterministic system is

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0022-4715/93/0800-0555$07.00/0

Hamiltonian (i.e., has no damping). As some damping is always present together with the noise in realistic systems (as quantified by the fluctuation-dissipation theorem in the case of systems in thermal equilibrium), the analysis of escape rates while neglecting damping is applicable only for times much smaller than the relaxation time.

The present study focuses on the tails of distribution, which are responsible for the particle escape to the distant boundaries. Mathematically, these tails can be described in the same way as in the weak-noise asymptotic (WNA) for conventional nonequilibrium systems (with damping),[2] as $\rho = Z \exp(-\phi/\eta)$, where Z and ϕ are both functions of the phase space variables and time, while η is a small general factor related to the diffusion intensity. The (time-dependent) escape rate r out of some boundary (from given initial conditions) can be found in the weak-noise approximation as $r = R \exp(-G/\eta)$, where G is the minimum of ϕ on the boundary.

In the present paper, the "tails" of the probability density distribution and the escape rates are studied for the Hamiltonian nonlinear oscillator with external noise. The Hamiltonian is assumed to be not exactly integrable, but only nearly so, i.e., consisting of an exactly integrable time-independent part and a small perturbation that may have a periodic dependence on time. Generically, such perturbations are known[3] to drive nonlinear resonances, which constitute an everywhere dense net. The essential question then is, how do the resonances affect the evolution of distribution tails and escape rates? We find that the situation is quite different for the 1D case (one spatial coordinate) and for higher dimensionalities. Indeed, in 1D the only spatial scale associated with each resonance is its width, which is proportional to the square root of the perturbation strength ε. It is quite natural therefore that the perturbation of ϕ by resonances, $\Delta\phi = \phi - \phi_0$ (the difference of ϕ's with and without resonances) in 1D should be of the order of $\sqrt{\varepsilon}$ and thus small for small ε.

In 2D and higher dimensions, the resonances are surfaces (or lines) in the action space, and the possibility of the particles to diffuse along them while staying inside the separatrices changes the situation drastically. The major reason for this is a certain "renormalization" of diffusion inside separatrices, leading to an increase of the component of diffusion intensity along the surface. This can be explained, in 2D for simplicity, as follows. In the plane of actions I_x, I_y, where resonances are lines, one can draw the arrow of separatrix oscillations, which shows the direction of trapped particle oscillations about the resonances line. The length of this arrow is the width of the separatrix (or twice the maximum oscillation amplitude) and its center is the resonance line (see Fig. 1). Now consider a small kick δ applied to a trapped particle in the direction orthogonal to the resonance line; it is clear that the center of oscillations will be displaced a distance

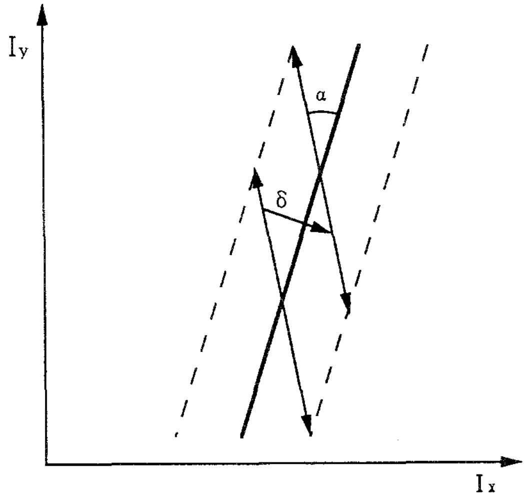

Fig. 1. The displacement of an inside-separatrix oscillation center by a transverse kick. The thick solid line is the resonance line. Dashed lines show the separatrix.

$\delta \cot(\alpha)$ along the resonance line. Similarly, if we introduce noise of intensity η in this direction, the diffusion of the oscillation center along the resonance will have the diffusion coefficient $\eta \cot(\alpha)$. Thus, for small angles α between the resonance oscillations and resonance line, diffusion is enhanced inside the separatrix. This enhancement was originally discovered and termed "resonance streaming" by Tennyson.[4] For escape rate and distribution tail problems, it leads to a very strong effect, since the particles can travel long distances (as compared to small resonance width) along the resonance lines. The decrease of the function ϕ by the effect of the resonances is of the order of unity (or $\phi - \phi_0 \sim \phi_0$) even for small perturbation strength ε as long as $\varepsilon \gg \eta$, which is a drastic exponentially strong amplification of the effect as compared to 1D. In this respect, the situation is quite similar to the escape rate and distribution function behavior in an oscillator with nonlinear resonances, damping, and noise,[5] where both damping and diffusion are "renormalized" within the separatrices.

In Section 2 we derive the average Fokker–Planck equation (FPE) in the vicinity of a nonlinear resonances, using the relative smallness of the diffusion intensity. In Section 3, we construct the WNA solution of that equation. As that solution is applicable only at short times, in Sections 4 and 5 we develop a modified WNA and its phenomenological solutions that are applicable at all times.

The general approach of the suggested theory is quite similar to that of ref. 5 for the steady-state distributions in the same type of system, but with damping. The present case of zero damping cannot be obtained, however, as the limiting case of that theory for vanishing damping, since the steady-state problem implies taking the limit $t \to \infty$ first.

2. LOCAL FPE

Our primary system of consideration is the two-dimensional Hamiltonian oscillator with external noise:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{p} \\ \dot{\mathbf{p}} &= -\frac{\partial U(\mathbf{x}, t)}{\partial \mathbf{x}} + (2\eta)^{1/2}\, \boldsymbol{\xi}(t) \end{aligned} \tag{1}$$

where η is the diffusion intensity. $\xi_i(t)$ here is the white-noise vector process $\langle \xi_i(t)\, \xi_k(t+\tau) \rangle = \delta_{ik}\, \delta(\tau)$. We suppose that the potential U consists of an unperturbed time-independent part $U_0(\mathbf{x})$, corresponding to exactly integrable motion, and a small perturbation $\varepsilon \delta U(\mathbf{x}, t)$, either time-independent or time-periodic with frequency Ω. The basic problem in which we are interested is how this small perturbation can affect the growth of distribution tails and escape rates. Escape rate processes come into play when the potential U_0 has several local minima or when an absorbing boundary is present in the $\mathbf{x}$ space.[1]

The Hamiltonian dynamics of the system (without noise) is nearly integrable for small ε. Following a standard approach, the Hamiltonian should be presented in action–angle variables of the unperturbed system:

$$H = H_0(\mathbf{I}) + \varepsilon \sum_{\boldsymbol{l}, n} V_{\boldsymbol{l}n}(\mathbf{I}) \cos(\boldsymbol{l} \cdot \boldsymbol{\theta} - n\Omega t) \tag{2}$$

where $H_0 = \mathbf{p}^2/2 + U_0(\mathbf{x})$ and the perturbation was expanded in Fourier series in both θ and t. Each harmonic $V_{\boldsymbol{l}n}$ excites a nonlinear resonance on the line $\boldsymbol{l} \cdot \mathbf{v}(\mathbf{I}) - n\Omega = 0$, where the frequence $\mathbf{v}$ is $\mathbf{v} = \partial H_0 / \partial \mathbf{I}$.[3] The amplitude of oscillation of $\mathbf{I}$ at the separatrix defines the "resonance width" ΔI in $\mathbf{I}$ space and is proportional to $\sqrt{\varepsilon}$ (see below). The resonant Hamiltonian can be obtained by dropping all the nonresonant harmonics, introducing new (canonical) variables

$$\begin{aligned} I_1 &= \frac{I_y}{l_y} \\ I_2 &= -I_x + \frac{l_x}{l_y} I_y \\ \psi_1 &= l_x \theta_x + l_y \theta_y - n\Omega t \\ \psi_2 &= -\theta_x \end{aligned} \tag{3}$$

and expanding the Hamiltonian $H_0(\mathbf{I})$ to second order in deviations in I_1 from the center $I_{10}(I_2)$.[3] The result is

$$H = \lambda \frac{p_1^2}{2} + \varepsilon V_{\mathbf{m}} \cos(\psi_1) \tag{4}$$

where $\lambda = \partial(l_x v_x + l_y v_y)/\partial I_1$ and $\mathbf{m} = (l, n)$. The resonance width, which is the amplitude of oscillations of p_1 on the separatrix of pendulum (4), is

$$p_{1r} = 2\left|\frac{\varepsilon V_{\mathbf{m}}}{\lambda}\right|^{1/2} \tag{5}$$

Now consider the effect of noise. The evolution of the distribution of particles, corresponding to the primary equations of motion (1), is governed by the FPE[6]

$$\frac{\partial \rho}{\partial t} + \mathbf{p}\frac{\partial \rho}{\partial \mathbf{x}_{\mathbf{M}}} - \frac{\partial(U_0 + \varepsilon \delta U)}{\partial \mathbf{x}}\frac{\partial \rho}{\partial \mathbf{p}} = \eta \frac{\partial^2 \rho}{\partial \mathbf{p}^2} \tag{6}$$

We will be constructing the solution of the FPE (6) under a set of limitations on the parameters. First, we will suppose that the Hamiltonian part of the dynamics can be well described in terms of isolated nonlinear resonances, so that the resonances do not overlap. This is true under the condition $(\varepsilon |V_{\mathbf{m}} \lambda|)^{1/2} \ll v_m$, where v_m is the smallest component of the frequency $\mathbf{v}$.[3] Second, we will be interested only in the tails of the distributions, which means that the characteristic energies $E = H_0$ and times of observation T should satisfy the condition $E/\eta T \gg 1$. The use of the WNA in Section 3 to describe the effect of nonlinear resonances is possible only under an extra condition of not too large a time t: $t \ll p_{1r} I/\eta Q$ (where Q and I are the characteristic values of the components of Q_{ij} and I_j). This condition was relinquished in Section 4, where the general case was considered, albeit through a phenomenological approach. Third, we suppose that the diffusion in our system is a slow process relative to both the unperturbed motion (time scale $\tau_1 \sim 1/v_m$) and the resonant oscillations [time scale $\tau_2 \sim (\varepsilon |V_{\mathbf{m}} \lambda|)^{-1/2}$]. The last condition is more restrictive and is obtained by requiring that the rms time τ required to shift the particle by diffusion to the distance equal to the resonance width (5), $\tau \sim p_{1r}^2/\eta$, is much larger than τ_2:

$$(\varepsilon |V_{\mathbf{m}}|)^{3/2} \gg \lambda^{1/2} \eta \tag{7}$$

This inequality, as well as the previous one, holds for small enough noise intensity η.

In the case of a slow diffusion, the distribution becomes constant along the Hamiltonian trajectories in a short time, allowing for the "thermal averaging" reduction of the FPE[7] (also known[6] as the fast variable elimination in the FPE in the more general context). In ref. 5, such averagings were performed in a similar FPE that included damping. Following this reference, we transform first the FPE (6) to the unperturbed action–angle variables $\mathbf{I}, \boldsymbol{\theta}$ and (supposing the distribution ρ depends only on ψ_1 and not on ψ_2) average it over both "fast" phases $\boldsymbol{\theta}$, keeping the "slow" phase ψ_1 contant. This yields

$$\frac{\partial \rho}{\partial t} + \varepsilon V_{\mathbf{m}} \sin \psi_1 \left(l_x \frac{\partial \rho}{\partial I_x} + l_y \frac{\partial \rho}{\partial I_y} \right) + (l_x v_x + l_y v_y - n\Omega) \frac{\partial \rho}{\partial \psi_1}$$
$$= \eta \frac{\partial}{\partial I_k} G_{0kl} \frac{\partial \rho}{\partial I_l} + \eta R_1 \frac{\partial \rho}{\partial \psi_1} + \eta R_2 \frac{\partial^2 \rho}{\partial \psi_1^2} + \eta R_{3k} \frac{\partial^2 \rho}{\partial I_k \, \partial \psi_1} \tag{8}$$

Here, only the resonant harmonic of the Fourier expansion in Eq. (2) was retained. The thermal-averaged diffusion tensor $G_{0kl}(I_x, I_y)$ is

$$G_{0kl} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_x \, d\theta_y \, \frac{\partial I_k(\mathbf{x}, \mathbf{p})}{\partial p_i} \frac{\partial I_l(\mathbf{x}, \mathbf{p})}{\partial p_i} \tag{9}$$

where the quantities under integration are expressed in I, θ variables after differentiation, and summation over the repeated indices is implied. The quantities R_1 through R_{3k} in (8) are other averages of the type (9) and are not given explicitly, since the corresponding terms drop out in subsequent transformations. Note also that the FPE (8) is local and is applicable only in the vicinity of a chosen resonance.

The FPE in the variables p_1, I_2, ψ_1 is the same as that of ref. 5 except for the damping terms, which are absent in our case:

$$\frac{\partial \rho}{\partial t} + \lambda p_1 \frac{\partial \rho}{\partial \psi_1} + \varepsilon V_{\mathbf{m}} \sin \psi_1 \frac{\partial \rho}{\partial p_1}$$
$$= \eta \left[Q_{11} \frac{\partial^2 \rho}{\partial p_1^2} + 2 Q_{21} \frac{\partial}{\partial p_1} \left(\frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right) \rho + Q_{22} \left(\frac{\partial}{\partial I_2} - \kappa \frac{\partial}{\partial p} \right)^2 \rho \right] + \eta P \tag{10}$$

where $\kappa(I_2) = \partial I_{10} / \partial I_2$ [$I_{10}(I_2)$ is the resonance line]. The quantity P contains a derivative of ρ with respect to ψ_1 that drops out later. The diffusion tensor Q_{ij} in (I_1, I_2) space is a linear transformation of the tensor G_{0kl} in I_x, I_y space. In the simpler case of uncoupled degrees of freedom in the unperturbed Hamiltonian H_0, the tensor G_{0kl} can be easily found[5] to be

$$G_{0kl} = \delta_{kl} \frac{I_k}{v_k} \tag{11}$$

where $v_k(I_k) = \partial H_0/\partial I_k$; there is no summation in k. The Q tensor is given then by

$$\begin{aligned}
Q_{11} &= \frac{I_1}{l_1 v_y} \\
Q_{21} &= \frac{l_x I_1}{l_y v_y} \\
Q_{22} &= \frac{(-I_2 + l_x I_1)}{v_x} + \frac{l_x^2}{l_y v_y} I_1
\end{aligned} \tag{12}$$

3. WEAK-NOISE ASYMPTOTICS

If all the parameters, including εV_m, are fixed while the noise intensity η is asymptotically small (so that $\eta Q \ll p_1, I$—see Section 4), the distribution function in the regions of the phase space where it was initially set zero can be presented in the standard WNA form:

$$\rho(p_1, I_2, \psi_1, t, \eta) = Z(p_1, I_2, \psi_1, t) \exp\left(-\frac{\phi(p_1, I_2, \psi_1, t)}{\eta}\right) \tag{13}$$

The escape rate r in the presence of absorbing boundary Γ somewhere in the initially unpopulated part of the phase space is known from the general WNA theory[2,6] to have the asymptotic form

$$r(t) = F(t) \exp\left(-\frac{R(t)}{\eta}\right) \tag{14}$$

where the quantity R is just the minimum of the function $\phi(\mathbf{x}, t)$ ($\mathbf{x}$ here is the phase space p_1, Ψ_1, I_2) on the boundary Γ:

$$R(t) = \min_{\mathbf{x} \in \Gamma} \phi(\mathbf{x}, t) \tag{15}$$

The partial differential equation for the function ϕ that defines the exponentially small factor in (13) is obtained from the FPE (10) by substituting the form (13) and singling out the highest powers of $1/\eta$, yielding

$$\frac{\partial\phi}{\partial t}+\lambda p_1\frac{\partial\phi}{\partial\psi_1}+\varepsilon V_{\mathrm{m}}\sin\psi_1\frac{\partial\phi}{\partial p_1}$$
$$=-Q_{11}\left(\frac{\partial\phi}{\partial p_1}\right)^2-2Q_{21}\frac{\partial\phi}{\partial p_1}\left(\frac{\partial\phi}{\partial I_2}-\kappa\frac{\partial\phi}{\partial p_1}\right)-Q_{22}\left(\frac{\partial\phi}{\partial I_2}-\kappa\frac{\partial\phi}{\partial p_1}\right)^2+\frac{\partial\phi}{\partial\psi_1}R \tag{16}$$

where R is some linear form in the gradients of ϕ and drops out later. This equation is the Hamilton–Jacobi equation (HJE) and can be solved through the method of characteristics. With that method, the solution with the initial condition $\phi=\infty$ in the initially unpopulated region of the phase space can be presented[2,6] in the variational form

$$\phi(\mathbf{x},t)=\min_{\tilde{x}(t)}\int_0^t d\tau\, L(\dot{\tilde{x}}(\tau),\tilde{x}(\tau)) \tag{17}$$

where the trajectories $\tilde{x}(\tau)$ start on the border of the unpopulated region and end at the point of observation $\tilde{x}(t)=\mathbf{x}$. The Lagrangian L in the expression (17) is related to the Hamiltonian H of the HJE (16) through the standard transformation

$$\begin{aligned}\tilde{p}_x&=\frac{\partial L(\tilde{x},\dot{\tilde{x}})}{\partial\dot{\tilde{x}}}\\ H(\tilde{x},\tilde{p}_x)&=\dot{\tilde{x}}\frac{\partial L}{\partial\dot{\tilde{x}}}-L\end{aligned} \tag{18}$$

The derivation of the HJE (16) was based on conventional techniques and did not contain any new approaches. The difficult part of the problem starts here and has to do with the nonperturbative nature of the dependence of the solution ϕ on the small parameter ε. Generally, one expects a discontinuity in this problem at $\varepsilon=0$, i.e.,

$$\phi_0(p_1,I_2,t)=\lim_{\varepsilon\to 0}\phi(p_1,I_2,\Psi_1,t,\varepsilon)\neq\phi_{\mathrm{un}}(p_1,I_2,t) \tag{19}$$

where ϕ_{un} is the unperturbed solution ϕ for $\varepsilon=0$. Mathematically, this discontinuity is associated with taking the limit $\eta\to 0$ before the limit $\varepsilon\to 0$. The physical explanation is that an arbitrarily small ε introduces a topologically distinct structure of the phase space (nonlinear resonance) that strongly affects the solutions of the HJE in spite of occupying a small $\sim\sqrt{\varepsilon}$ phase-space volume. More specifically, the diffusion acts quite differently inside the separatrix of the resonance; the phenomenon was originally described from a single-particle perspective and is called "resonance streaming."[4] In systems with damping, one observes a similar amplification of the effect of perturbation on the escape rates in the many-dimensional case.[5] There, it can be viewed as a stronger and qualitatively different type of "stochastic resonance" effects in many-dimensional systems as compared to the conventionally discussed one-dimensional case (see, e.g., ref. 8).

We will be calculating the leading-order quantity ϕ_0 through what can be viewed as a version of singular perturbation theory. Notice that as $\varphi = \phi_0 t$ is independent of the phase ψ_1 and therefore time t, all relevant quantities (distribution function, escape rates) are singularly strongly perturbed and in the leading logarithmic approximation have a simple scaling with time. Similar to the normal nonsingular periodic modulation of escape rates and distribution functions in overdamped systems with periodic driving,[8] our system also has such modulations, but they can be neglected as a higher-order effect.

The calculation of the singular-limit function ϕ_0 will be based on the empirical assumption that one possible class of solutions of the HJE (16) has the following leading-order (in powers of ε) behavior inside the separatix of the resonance:

$$\phi(I_2, p_1, \psi_1, t) = \phi_0(I_2, t) + R(I_2, t)\, H(p_1, I_2, \psi_1) \tag{20}$$

where H is given by Eq. (4). More consistent, but much more tedious calculations can be carried out based on the variational representation (17) along the same lines as in ref. 5 for a similar system with damping. Plugging the solution (20) in Eq. (16) and isolating the lowest-order terms in ε, one obtains

$$\frac{\partial \phi_0}{\partial t} = -Q_{22} \left(\frac{\partial \phi_0}{\partial I_2} \right)^2 \tag{21}$$

Everywhere outside of the separatrix, the function ϕ_0 quite naturally obeys the unperturbed ($V_m = 0$) HJE (16), which is more natural to present in the original variables I_x, I_y:

$$\frac{\partial \phi_0}{\partial t} = -G_{0kl} \frac{\partial \phi_0}{\partial I_l} \frac{\partial \phi_0}{\partial I_k} \tag{22}$$

Thus, the evolution of ϕ_0 in the $\mathbf{I}$ plane has to obey Eq. (22) with the boundary condition on a section of the resonance line $I_{10}(I_2)$ that is defined by Eq. (21). The section of the line that provides the boundary condition has to be found self-consistently from the condition of minimizing the values of ϕ_0 on the line. Recall that we are considering the growth of tails in the initially unpopulated areas ($\rho = 0$, $\phi = \infty$). The time dependence of ϕ_0 that satisfies these initial conditions and both HJEs (21) and (22) is $\phi_0(\mathbf{I}, t) = \varphi(\mathbf{I})/t$. Note also that the same time dependence is true for the unperturbed solution $\phi_{\text{un}}(\mathbf{I}, t) = \varphi_{\text{un}}(\mathbf{I})/t$. Then, the variation of φ along the section of the resonance line is found to be

$$\varphi_r(I_2) = \left\{ [\varphi_{\text{un}}(I_s)]^{1/2} + \int_{I_s}^{I_2} \frac{dI_2'}{[Q_{22}(I_2')]^{1/2}} \right\}^2 \tag{23}$$

where $I_2 = I_s$ demarcates the point between the "active" section of the resonance [where φ is defined by expression (23)] and the "passive" one [where φ is unperturbed, $\varphi = \varphi_{\text{un}}(\mathbf{I})$].

The quantity $I_2 = I_s$ from the above-formulated minimization condition is the solution of the equation

$$\left| \frac{d\varphi_{\text{un}}(I_{10}(I_2), I_2)}{dI_2} \right| = \left[\frac{\varphi_{\text{un}}(I_{10}(I_2), I_2)}{Q_{22}(I_2)} \right]^{1/2} \tag{24}$$

From the same argument, one obtains the condition of the strong (singular) perturbations $\varphi \neq \varphi_{\text{un}}$ by the resonance in the form of the inequality

$$\left| \frac{d\varphi_{\text{un}}(I_{10}(I_2), I_2)}{dI_2} \right| \geqslant \left[\frac{\varphi_{\text{un}}(I_{10}(I_2), I_2)}{Q_{22}(I_2)} \right]^{1/2} \tag{25}$$

which has to be satisfied on some part of the resonance line.

In the $\mathbf{I}$ plane, the equation for the function φ is obtained from the unperturbed HJE (21) to be

$$\varphi = G_{0kl} \frac{\partial \varphi}{\partial I_l} \frac{\partial \varphi}{\partial I_k} \tag{26}$$

Now, one can construct the function φ from the minimization principle (17) by introducing the function $\tilde{\varphi}(\mathbf{I})$ that is the continuously differentiable solution of the unperturbed HJE (26) subject to the boundary condition (23). The function φ is implicitly defined by these boundary conditions through the standard characteristics method solution and therefore will be considered as known. The minimization then reduces to taking the minimum between the unperturbed function φ_{un} and the function $\tilde{\varphi}$:

$$\varphi(\mathbf{I}) = \min(\varphi_{\text{un}}(\mathbf{I}), \tilde{\varphi}(\mathbf{I})) \tag{27}$$

This is our final expression for the logarithmic asymptotics of the distribution tails $\varphi = \lim_{\eta \to 0} \eta t \ln(\rho)$. The function φ defined in this way is continuously differentiable in separate regions of the plane **I** where $\varphi = \varphi_{\text{un}}$ or $\varphi = \tilde{\varphi}$, with the discontinuities in the derivatives on the boundaries of the regions. The discontinuities in the derivatives are a generic feature of the WNA in dissipative systems in general[9] and are present as well in the equivalent of our system with damping.[10]

The formulas (23) and (27) can be easily generalized to include the case of several "active" sections on one resonance line and/or several resonance lines. That generalization can be done similar to the approach of ref. 5 for a like system with damping. Applying formula (15) in the leading logarithmic approximation, the escape rate r is defined by the minimum of the function φ of (27) on the absorbing boundary $r \sim \exp(-\varphi_{\min}/\eta t)$.

4. MODIFIED WEAK-NOISE ASYMPTOTICS

The WNA analysis of Section 3 is limited by the condition of not too large a time t, since the characteristic random shift $\Delta I = (\eta Q t)^{1/2}$ has to be less than the relevant smallest relevant dynamical scale of the problem. This scale is associated with the small resonance width and turns out (see below) to be $(I p_{1r})^{1/2}$, so that the restriction on time t is $t \ll p_{1t} I/\eta Q$. Since it takes much longer, $t_1 \sim I^2/\eta Q$, for the particle to diffuse a distance I (escape), the problem of the tail growth at times $I^2/\eta Q \gg t \gg p_{1r} I/\eta Q$, when the WNA breaks down on the small scale of the resonance width, is relevant and will be addressed in this section.

The breakdown of the WNA occurs only in the vicinity of the resonance $I_1 - I_{10}(I_2) \sim p_{1r}$. Therefore, in that vicinity we present the distribution function ρ in the limit of small noise $\eta \to 0$ and small resonance width $p_{1r} \sim \eta$ in the partially asymptotic form

$$\rho(p_1, I_2, \psi_1, t, \eta) = Z(p_1, I_2, \psi_1, t, \eta) \exp\left(-\frac{\phi(I_2, t)}{\eta}\right) \tag{28}$$

Here, the variation along the resonance line has a normal exponential form $[(1/Z)\,\partial Z/\partial I_2 \to \text{const},\ (1/Z)\,\partial Z/\partial t \to \text{const for } \eta \to 0]$, while the dependence on transverse variables p_1, Ψ was left unrestricted. The method of using this generalized WNA was proposed in ref. 7 for a system with damping. Substituting expression (28) in the FPE (10) and singling out the highest degrees of $1/\eta$, one arrives at

$$\tilde{L}_H Z = \left(\frac{a}{\eta} + b\frac{\partial}{\partial p_1} + c\eta\frac{\partial^2}{\partial p_1^2}\right) Z + \eta F_1 \frac{\partial^2 Z}{\partial p_1\, \partial \psi_1} + F_2 q \frac{\partial Z}{\partial \psi_1} \tag{29}$$

where $q = \partial\phi/\partial I_2$, $\tilde{L}_H$ is the Liouville operator

$$\tilde{L}_H = \frac{\partial H}{\partial p_1}\frac{\partial}{\partial \psi_1} - \frac{\partial H}{\partial \psi_1}\frac{\partial}{\partial p_1}$$

of the resonant Hamiltonian (4). The notations a, b, c were introduced for

$$\begin{aligned} a(I_2, t) &= Q_{22}\left(\frac{\partial \phi}{\partial I_2}\right)^2 + \frac{\partial \phi}{\partial t} \\ b(I_2, t) &= 2(\kappa Q_{22} - Q_{21})\frac{\partial \phi}{\partial I_2} \\ c(I_2) &= Q_{11} + Q_{22}\kappa^2 - 2Q_{21}\kappa \end{aligned} \tag{30}$$

where all Q_{ij} generally depend on I_2. Note that the derivative $\partial Z/\partial t$ does not enter Eq. (29). The quantities F_1, F_2 originate from the last term in the FPE (10) and will drop out in subsequent calculations.

Utilizing condition (7) of diffusion slowness relative to the resonant dynamics, one can now perform the second stage of thermal averaging along the trajectories of the Hamiltonian H, Eq. (4). The procedure is the same as in ref. 5—we suppose that the function Z depends on p_1 and ψ_1 only through the action $J(H)$ for the Hamiltonian H of (4) and average Eq. (29) over time. The resulting equation is

$$\frac{a}{\eta} Z + \frac{\partial}{\partial J}\left(bF + c\eta G(J)\frac{\partial}{\partial J}\right) Z = 0 \tag{31}$$

where

$$\begin{aligned} F(J) &= \left\langle \frac{\partial J(p_1, \psi_1)}{\partial p_1} \right\rangle \\ G(J) &= \left\langle \left(\frac{\partial J(p_1, \psi_1)}{\partial p_1}\right)^2 \right\rangle \end{aligned} \tag{32}$$

The symbol $\langle \cdots \rangle$ here implies the averaging over time along the trajectories of the Hamiltonian H. Notice also that the last two terms in Eq. (29) vanished under the averaging.

A specific class of solutions of Eq. (31) is defined by the conditions of the function $Z(J(H))$ to have a maximum at the center of the resonance $H = -\varepsilon|V_m|$ and going to zero for $H \to \infty$ both above and below the resonance. The requirement of compatibility of these conditions gives a relation between the constants a, b, and c that is the HJE for ϕ. Technically, however, Eq. (31) is intractable, since the quantities F and G are expressed through elliptic integrals.[7] One way of handling this problem is the phenomenological simplification of functions $F(J)$ and $G(J)$ as suggested in ref. 10.

5. PHENOMENOLOGICAL APPROACH

Following the approach of ref. 10, we simplify the functions $F(J)$ and $G(J)$ by substituting the exact trajectories of the pendulum (4) by simplified trajectories shown in Fig. 2, and subsequent averaging in Eq. (29) along these trajectories.

Equation (29) is formally the same as that of ref. 10, while the difference is in the definition of coefficients a and b in (30). Therefore we can use all the intermediate calculations from ref. 10. The resulting one-dimensional HJE, defining the relation between $\partial\phi/\partial t$ and $\partial\phi/\partial I_2$, is Eq. (32) of ref. 10, which reads

$$\left[\frac{2}{ac}(b^2-4ac)\right]^{1/2} = \tan\left[\left(2\frac{a}{c}\right)^{1/2}\frac{kp_{1r}}{\eta}\right] \tag{33}$$

where k is a phenomenological constant of the order of unity. For the regime with large resonance width p_{1r} and small noise η one recovers the WNA result, since the solution $a=0$ of Eq. (33) is the same as Eq. (21).

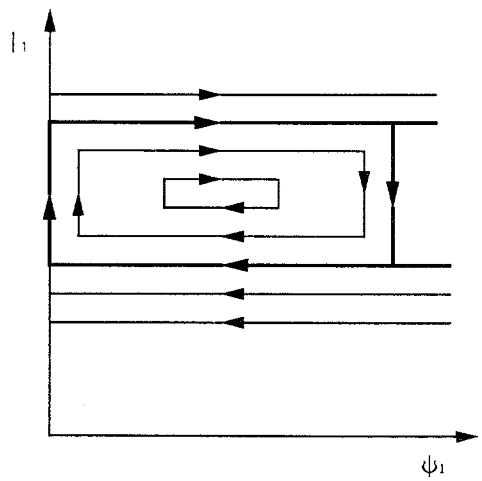

Fig. 2. "Simplified" trajectories approximating pendulum trajectories of the nonlinear resonance.

The condition of applicability of this solution is obtained by estimating $a \sim \partial\phi/\partial t \sim \Delta I^2/Qt^2$ to yield

$$\frac{p_{1r}}{\eta} \gg \frac{Qt}{|I_{21} - I_{20}|} \tag{34}$$

where Q is the characteristic value of the components of Q_{ij} and t is the time allowed for the transition alonog the line from $I_2 = I_{20}$ to $I_2 = I_{20} + \Delta I$.

Equation (33) is the one-dimensional HJE for the evolution of ϕ on some self-consistently chosen section of the resonance line, as it implicitly defines $\partial\phi/\partial t$ as a function of $\partial\phi/\partial I_2$:

$$\frac{\partial\phi}{\partial t} = G\left(I_2, \frac{\partial\phi}{\partial I_2}\right) \tag{35}$$

To gain a qualitative insight into the nature of this equation, one can use the condition that the arguments of the square roots in both the rhs and lhs of Eq. (33) are positive, yielding the inequality

$$Q_{22}\left(\frac{\partial\phi}{\partial I_2}\right)^2 < \left|\frac{\partial\phi}{\partial t}\right| < Q'\left(\frac{\partial\phi}{\partial I_2}\right)^2 \tag{36}$$

where

$$Q' = \frac{Q_{11}Q_{22} - Q_{21}^2}{Q_{11} - 2\kappa Q_{21} + \kappa^2 Q_{22}}$$

This inequality is indicative of the crossover of the solution of Eq. (32) from a small resonance width p_{1r} to a large one. Indeed, for small p_{1r} and large transition time t it becomes $b^2 - 4ac = 0$, or $\partial\phi/\partial t = -Q'(\partial\phi/\partial I_2)^2$; while for large p_{1r} and small transition time t, it becomes $a = 0$, or $\partial\phi/\partial t = -Q_{22}(\partial\phi/\partial I_2)^2$. Thus, for the effective diffusion intensity along the line one has a crossover from the value $\eta Q'$ in the former case to the value ηQ_{22} in the latter [note that the condition $Q' > Q_{22}$ is weaker than the necessary condition (25)]. The meaning of the quantity Q' is that of an efective diffusion along the resonance line in the absence of resonance, defined through the WNA of the transition probability between two close points on that line.

As with the WNA, one has to solve the one-dimensional HJE (34) in conjunction with the unperturbed HJE (21) in the **I** plane. Again, a self-consistently found section of the resonance line will provide the boundary condition for the HJE in the plane. The procedure will be more involved, however, since there is no simple time dependence $\phi = \varphi/t$. Similarly to the WNA case of Section 3, one can proceed from the minimization principle of the type (17) by using it in its "reduced" two-dimensional form. Then, the characteristics $\mathbf{I}(\tau)$ have the Lagrangian L corresponding to the unperturbed HJE (20), while the characteristics going along some section of the resonance line have the one-dimensional Lagrangian corresponding to the HJE (35). The "active" section of the resonance line and the value of ϕ on it have to be found from the minimization condition

$$\phi_r(I_2, t) = \min_{t_1, I_s} \left(\frac{\varphi_{\rm un}(I_s)}{t - t_1} + \bar{\phi}_r(I_s, I_2, t_1) \right) \tag{37}$$

where the first term is the contribution from the unperturbed characteristic coming from the initially populated region to the point on the resonance line $I_2 = I_s$, and the second is the contribution from the characteristic along the resonance line. Notice here that unlike the WNA case, the "active" section changes in time, as I_s will depend on t. The quantity $\bar{\phi}_r$ is defined as the solution of the one-dimensional HJE (35) with the initial and boundary conditions $\bar{\phi}_r(I_s, I_2, t=0) = \infty$, $\bar{\phi}_r(I_s, I_s, t) = 0$, and is defined implicitly through the standard characteristics method. For the purpose of illustration, consider the case when Q_{ij} are constant, so that the function G in Eq. (35) does not depend on I_2. The function $\bar{\phi}_r$ then is found to be

$$\bar{\phi}_r(I_s, I_2, t) = p(I_2 - I_s) + tG(p) \tag{38}$$

where p is the solution of the equation

$$I_2 - I_s = -t \frac{dG}{dp}(p) \tag{39}$$

One can introduce now the function $\tilde{\phi}$ that is implicitly defined as the characteristics method solution of the unperturbed HJE (21) subject to the boundary condition (37). The final expression for the function ϕ in the plane $\mathbf{I}$ is similar to formula (27) of Section 3:

$$\phi(\mathbf{I}, t) = \min \left(\frac{\varphi_{\rm un}}{t}, \tilde{\phi}(\mathbf{I}, t) \right) \tag{40}$$

6. DISCUSSION AND CONCLUSIONS

We described the growth of the distribution tails in a two-dimensional Hamiltonian system with external noise. This growth can be strongly accelerated due to an enhanced rate of diffusive transport of particles along the resonances. The basic scenario of diffusion enhancement inside the separatrices in the case of a small angle between the resonance line and the resonance oscillations direction was previously described in the literature and termed "resonance streaming."[4] However, the single-particle calculation of "renormalized" diffusion intensity inside the separatrix[4] does not provide the essential information about the macroscopic transport rate along the resonance and its overall effect on the distribution function. Consistent treatment of that kind is possible in the framework of the WNA, with a result that for short observation times confirms the single-particle estimate.[4] For longer times, when the deviation of the particle on the scale of a resonance width becomes probable, the situation is more complex. The basic dynamic process that accounts in this case for the "macroscopic" transport rate (i.e., observed on a scale much larger than the resonance width) is the diffusion of particles in the direction transverse to the resonance line. Because of the different longitudinal diffusion intensities inside and outside of the separatix, the transverse diffusion modulates the longitudinal one. This makes the effective one-dimensional random walk along the resonance line a more complex stochastic process, analyzed with the use of some phenomenological simplifications.

In most physical processes the noise is coming from an interaction with a certain heat bath and is always accompanied by a certain amount of damping. Then, the present theory is applicable only when the damping can be neglected. This leads to the requirement of the observation time to be much less than the damping time. In the opposite case of much larger times one has a more conventional situation of a time-independent escape rate, and the effect of nonlinear resonances in such a system was analyzed in related work.[5,10]

ACKNOWLEDGMENT

The Fermi National Accelerator Laboratory is operated by the Universities Research Association under contract with the U.S. Department of Energy.

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